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Electromagnetic scattering by dielectric spheroids in the forward and backward directions

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Abstract. It is shown that expressions, giving the forward and backward scattering amplitudes for the scattering of electromagnetic waves by arbitrarily orientated spheroids in terms of the respective amplitudes for incidence along the principal axes, hold under various approximations. The Van der Hulst approximation for spheroids is derived.

1. Introduction

Information on the scattering of electromagnetic waves by spheroidal scatterers is needed in many disciplines. Knowledge of scattering amplitudes in the forward and backward directions, for arbitrary scatterer orientation, is particularly required in, for example, radiowave propagation and radar. Since scattering amplitude calculations, though now possible (Asano and Yamamoto 1975, Holt *et al* 1976), can be time consuming, it is desirable to be able to obtain amplitudes for arbitrary scatterer orientation in terms of amplitudes for incidence along the principal axes. Warner (1975) proposed approximate expressions for the backscattering and extinction cross sections for spheroids at any incidence angle in terms of those for 0° and 90° incidence. Subsequently, but independently, Uzunoglu *et al* (1976) proposed a similar expression for the forward scattering amplitudes. These expressions were obtained empirically. In this paper we seek to put the scattering amplitude expression on a firmer theoretical basis and to show that, under restricted conditions, it will also apply to backscattering amplitudes. In the course of this we derive the expression for the scattering amplitude in the forward direction for scattering by a spheroid under the approximation of Van der Hulst (1957) and compare it with the Rayleigh–Gans approximation.

2. Theory

We consider the scattering of a plane electromagnetic wave incident on a dielectric spheroid of semi-axes a, a, c and refractive n_0. We assume the incident direction, $\hat{k}$, to make an angle θ with the axis of symmetry of the spheroid, which we take as the z axis. The x axis is taken to lie in the plane containing the z axis and $\hat{k}$. We define the polarisation to be either vertical (V) or horizontal (H) according as the electric vector either lies in or is perpendicular to the xz plane. We define $f_{\mathrm{V,H}}(\theta, {}^{0}_{\pi})$ to be the scattering amplitude for a vertically (horizontally) polarised wave, incident at an angle θ

to the axis of symmetry, to be scattered, with no change in polarisation, in the forward or backward direction. Uzunoglu *et al* (1976) proposed the following approximation linking the forward amplitude at incident angle θ to those for incidence along the principal axes:

$$f_{\mathrm{V,H}}(\theta, 0) = f_{\mathrm{V,H}}(0, 0) \cos^2 \theta + f_{\mathrm{V,H}}(\pi/2, 0) \sin^2 \theta. \tag{1}$$

This expression was based on an analysis of scattering amplitudes for scattering of microwaves of frequencies in the region 4–30 GHz by raindrops.

We investigate the validity of this expression, and a similar expression for the backward direction

$$f_{\mathrm{V,H}}(\theta, \pi) = f_{\mathrm{V,H}}(0, \pi) \cos^2 \theta + f_{\mathrm{V,H}}(\pi/2, \pi) \sin^2 \theta \tag{2}$$

by considering the scattering under several approximations—the Rayleigh, Rayleigh–Gans, and Van der Hulst approximations.

It is worthwhile pointing out that expressions (1) and (2) are correct in the limit of a sphere, for then the scattering amplitude is independent of θ.

2.1. Rayleigh scattering

We assume that $k_0 a \ll 1$ and that $|n_0| k_0 a \ll 1$. The scattering amplitude is given by (Van der Hulst 1957)

$$f(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{e}) = (\mathbf{A}\boldsymbol{e}).\boldsymbol{e} - (\hat{\boldsymbol{k}}'.\boldsymbol{e})[(\mathbf{A}\boldsymbol{e}).\boldsymbol{e}] \tag{3}$$

where $\boldsymbol{k}$, $\boldsymbol{k}'$ are the incident and scattered wave vectors and $\boldsymbol{e}$ is the incident polarisation. $\mathbf{A}$ is a tensor, and since we take the axes to be the principal axes of the scatterer, $\mathbf{A}$ is diagonal. Since $\boldsymbol{k}\,.\,\boldsymbol{e} = 0$,

$$f(\boldsymbol{k}, \boldsymbol{k}, \boldsymbol{e}) = (\mathbf{A}\boldsymbol{e}).\boldsymbol{e} = f(\boldsymbol{k}, -\boldsymbol{k}, \boldsymbol{e}) \tag{4}$$

and thus, as is well known, the forward and backward scattering amplitudes are equal. Thus if equation (1) is valid, equation (2) will also be valid.

Now

$$\hat{\boldsymbol{k}} = \sin \theta\, \hat{\boldsymbol{x}} + \cos \theta\, \hat{\boldsymbol{z}} \tag{5}$$

For horizontal polarisation, therefore, $\boldsymbol{e} = \hat{\boldsymbol{y}}$, hence

$$f_{\mathrm{H}}(\theta, 0) = f(\boldsymbol{k}, \boldsymbol{k}, \hat{\boldsymbol{y}}) = A_2.$$

Thus $f_{\mathrm{H}}(\theta, 0)$ is independent of θ and consequently equation (1) is satisfied.

For vertical polarisation, $\boldsymbol{e} = -\cos \theta\, \hat{\boldsymbol{x}} + \sin \theta\, \hat{\boldsymbol{z}}$ and thus

$$\begin{aligned} f_{\mathrm{V}}(\theta, 0) &= (-A_1 \cos \theta\, \hat{\boldsymbol{x}} + A_3 \sin \theta\, \hat{\boldsymbol{z}})\,.\,(-\cos \theta\, \hat{\boldsymbol{x}} + \sin \theta\, \hat{\boldsymbol{z}}) \\ &= A_1 \cos^2 \theta + A_3 \sin^2 \theta. \end{aligned} \tag{6}$$

Thus $f_{\mathrm{V}}(0, 0) = A_1$ and $f_{\mathrm{V}}(\pi/2, 0) = A_3$ and hence equation (1) is again satisfied. Thus for both polarisations, equations (1) and (2) are satisfied for Rayleigh scattering.

2.2. Rayleigh–Gans theory

In the Rayleigh–Gans approximation we assume that the refractive index is close to 1, i.e. $|n_0 - 1| \ll 1$. The body is then considered to consist of volume elements which are spherical Rayleigh scatterers. Consequently there is no distinction between the two incident polarisations. The scattering amplitude is given by (Newton 1966)

$$f(\boldsymbol{k}, \boldsymbol{k}') = \frac{k^2}{4\pi} \int_V (n_0^2 - 1) \exp[\mathrm{i}(\boldsymbol{k} - \boldsymbol{k}') \cdot \boldsymbol{r}] \, \mathrm{d}\boldsymbol{r} \tag{7}$$

where V is the volume of the scatterer.

For a spheroid, straightforward analysis yields

$$f(\boldsymbol{k}, \boldsymbol{k}') = a^2 c k^2 (n_0^2 - 1) j_1(X)/X \tag{8}$$

where $j_n(X)$ is the spherical Bessel function of order n and

$$X^2 = k^2(a^2 \sin^2\theta_k + c^2 \cos^2\theta_k) + k'^2(a^2 \sin^2\theta_{k'} + c^2 \cos^2\theta_{k'}) - 2kk'[c^2 \cos\theta_k \cos\theta_{k'} + a^2 \sin\theta_k \sin\theta_{k'} \cos(\phi_k - \phi_{k'})]. \tag{9}$$

In powers of X, we have

$$j_1(X)/X = \tfrac{1}{3} - \tfrac{1}{30}X^2 + \tfrac{1}{840}X^4 - \cdots. \tag{10}$$

Hence, in the forward direction where $X = 0$, $f(\boldsymbol{k}, \boldsymbol{k}')$ is independent of incident direction and thus equation (1) holds.

In the backward direction

$$X = 2k(a^2 \sin^2\theta + c^2 \cos^2\theta)^{1/2}. \tag{11}$$

Since $f(\pi/2, \pi) = j_1(2ka)/2ka$ and $f(0, \pi) = j_1(2kc)/2kc$, if only terms to order X^2 are taken in (10), equation (2) can be seen to be satisfied exactly. The error in equation (2) could thus only become apparent when $j_1(X)/X$ differs significantly from the first two terms in its series expansion—i.e. $ka \gtrsim 1{\cdot}8$. Consequently it is clear that the applicability of expression (2) is more restricted than that for the forward direction.

2.3. *Van der Hulst scattering*

Van der Hulst (1957) introduced this approximation to investigate the forward scattering amplitude when $|n_0 - 1| \ll 1$, but when $k_0 a$ is allowed to be large. We shall assume that the scatterer is a general ellipsoid of semi-axes a, b, c.

The forward amplitude is given by (Newton 1966)

$$f(0) \simeq \frac{k}{2\pi\mathrm{i}} \int \mathrm{d}S\{\exp[\mathrm{i}k(n_0 - 1)\delta] - 1\} \tag{12}$$

where δ is the path length (assumed undeviated) of a ray parallel to the incident direction and $\mathrm{d}S$ is an element of the cross-sectional area of the ellipsoid perpendicular to the incident direction (see figure 1). Referred to its principal axes, the ellipsoid may

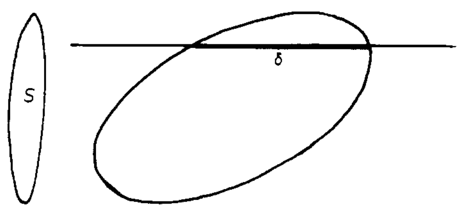

Figure 1. Path length δ in the ellipsoid and cross-sectional area S in the Van der Hulst approximation.

be written as

$$\boldsymbol{u}^{\mathrm{T}}\mathbf{D}\boldsymbol{u} = 1,$$

where

$$\boldsymbol{u}^{\mathrm{T}} = (x, y, z) \qquad \text{and} \qquad \mathbf{D} = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{pmatrix}.$$

The unit vector parallel to the incident direction we shall take to be $\hat{\boldsymbol{k}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ where ϕ is now the azimuthal angle.

If we now consider the coordinate transformation $\boldsymbol{u}' = \mathbf{A}\boldsymbol{u}$ where

$$\mathbf{A} = \begin{pmatrix} -\sin\psi\sin\phi - \cos\theta\cos\phi\cos\psi, & \sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi, & \cos\psi\sin\theta \\ \cos\psi\sin\phi - \cos\theta\cos\phi\sin\psi, & -\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi, & \sin\psi\sin\theta \\ \sin\theta\cos\phi, & \sin\theta\sin\phi, & \cos\theta \end{pmatrix}$$

(cf Goldstein 1959) (13)

the incident direction becomes the z' axis. Note that $\mathbf{A}$ is orthogonal. The equation of the ellipse is now

$$\boldsymbol{u}'^{\mathrm{T}}\mathbf{C}\boldsymbol{u}' = 1, \qquad \text{where } \mathbf{C} = \mathbf{ADA}^{\mathrm{T}} \tag{14}$$

Consider now vectors of the form $\boldsymbol{u} = \boldsymbol{s} + \lambda\hat{\boldsymbol{k}}$ where $\boldsymbol{s}\cdot\boldsymbol{k} = 0$. They intersect the ellipsoid where $(\boldsymbol{s}+\lambda\hat{\boldsymbol{k}})^{\mathrm{T}}\mathbf{C}(\boldsymbol{s}+\lambda\hat{\boldsymbol{k}}) = 1$ or

$$\lambda^2\hat{\boldsymbol{k}}^{\mathrm{T}}\mathbf{C}\hat{\boldsymbol{k}} + 2\lambda\hat{\boldsymbol{k}}^{\mathrm{T}}\mathbf{C}\boldsymbol{s} + \boldsymbol{s}^{\mathrm{T}}\mathbf{C}\boldsymbol{s} - 1 = 0. \tag{15}$$

The path length in the ellipsoid is $|\lambda_1 - \lambda_2|$, where λ_1, λ_2 are the roots of (15) and hence

$$\delta = \frac{2}{\hat{\boldsymbol{k}}^{\mathrm{T}}\mathbf{C}\hat{\boldsymbol{k}}}[(\hat{\boldsymbol{k}}^{\mathrm{T}}\mathbf{C}\boldsymbol{s})^2 - \hat{\boldsymbol{k}}^{\mathrm{T}}\mathbf{C}\hat{\boldsymbol{k}}(\boldsymbol{s}^{\mathrm{T}}\mathbf{C}\boldsymbol{s} - 1)]^{1/2} \tag{16}$$

The boundary of S is given by the requirement that $\delta = 0$, and hence satisfies

$$(\hat{\boldsymbol{k}}^{\mathrm{T}}\mathbf{C}\boldsymbol{s})^2 = \hat{\boldsymbol{k}}^{\mathrm{T}}\mathbf{C}\hat{\boldsymbol{k}}[\boldsymbol{s}^{\mathrm{T}}\mathbf{C}\boldsymbol{s} - 1]. \tag{17}$$

Assuming that $\mathbf{C} = [c_{ij}]$ and $\boldsymbol{s}^{\mathrm{T}} = (s_1, s_2, 0)$, (17) becomes

$$\alpha^2 s_1^2 + 2\gamma s_1 s_2 + \beta^2 s_2^2 = 1 \tag{18}$$

where

$$\alpha^2 = \frac{c_{11}c_{33} - c_{13}^2}{c_{33}} \qquad \beta^2 = \frac{c_{22}c_{33} - c_{23}^2}{c_{33}} \qquad \gamma = \frac{c_{12}c_{33} - c_{13}c_{23}}{c_{33}}. \tag{19}$$

So far, ψ has been arbitrary. We now choose ψ such that $\gamma = 0$ (which simply chooses the x', y' axes to be the principal axes of S). Hence from (12)

$$f(0) \approx \frac{k}{2\pi\mathrm{i}}\int_{\alpha^2 s_1^2 + \beta^2 s_2^2 = 1} \mathrm{d}S\left[\exp\left(\frac{2\mathrm{i}k(n-1)}{c_{33}^{1/2}}(1 - \alpha^2 s_1^2 - \beta^2 s_2^2)^{1/2}\right) - 1\right] \tag{20}$$

Putting

$$s_1=\frac{t\cos\tilde{\phi}}{\alpha},\qquad s_2=\frac{t\sin\tilde{\phi}}{\beta},\qquad \mu=\frac{2k(n_0-1)}{c_{33}^{1/2}},$$

(20) becomes

$$f(0)\simeq\frac{k}{2\pi\mathrm{i}\alpha\beta}\int_0^1 t\,\mathrm{d}t\,[\exp(\mathrm{i}\mu(1-t^2)^{1/2}-1]\int_0^{2\pi}\mathrm{d}\tilde{\phi} \tag{21}$$

which yields

$$f(0)=\frac{\mathrm{i}k}{\mu^2\alpha\beta}[1+\tfrac{1}{2}\mu^2-\exp(\mathrm{i}\mu)\{1-\mathrm{i}\mu\}]. \tag{22}$$

From (13), (14) we see

$$c_{33}=\frac{\sin^2\theta\cos^2\phi}{a^2}+\frac{\sin^2\theta\sin^2\phi}{b^2}+\frac{\cos^2\theta}{c^2} \tag{23}$$

and thus for a spheroid

$$\mu=\frac{2k(n_0-1)}{(\sin^2\theta/a^2+\cos^2\theta/c^2)^{1/2}} \tag{24}$$

To calculate α, β notice that α^2, β^2, γ are related to elements of $\mathbf{C}^{-1}$, and since $\mathbf{A}$ is orthogonal,

$$\mathbf{C}^{-1}=\mathbf{A}\mathbf{D}^{-1}\mathbf{A}^{\mathrm{T}}. \tag{25}$$

For a spheroid, the condition $\gamma=0$ is equivalent to $\psi=0$ and we obtain

$$\alpha=(acc_{33}^{1/2})^{-1}\qquad \beta=a^{-1} \tag{26}$$

Thus

$$f(\theta,0)=(\mathrm{i}ka^2cc_{33}^{1/2}/\mu^2)[1+\tfrac{1}{2}\mu^2-\mathrm{e}^{\mathrm{i}\mu}(1-\mathrm{i}\mu)]. \tag{27}$$

For small μ

$$1+\tfrac{1}{2}\mu^2-\mathrm{e}^{\mathrm{i}\mu}(1-\mathrm{i}\mu)\simeq-\mu^3/3+\mu^4/8 \tag{28}$$

and thus

$$f(0)\sim\tfrac{1}{3}ka^2cc_{33}^{1/2}\mu=\tfrac{2}{3}k^2(n_0-1)a^2c \tag{29}$$

which is consistent with the Rayleigh–Gans expression since $|n_0-1|\ll 1$. For large μ,

$$f(\theta,0)\sim\tfrac{1}{2}\mathrm{i}ka^2c(\sin^2\theta/a^2+\cos^2\theta/c^2)^{1/2}+\mathrm{O}(1/\mu). \tag{30}$$

It may be seen that (30) satisfies (1) in the limit of small eccentricities (i.e. $c/a\sim 1$).

3. Discussion

In the preceding sections we have shown that the forward scattering rule (1) is satisfied exactly under the Rayleigh and Rayleigh–Gans approximations, and also for ka large provided that both the refractive index and the eccentricity are close to unity. The range of validity of the backward scattering rule (2) appears more restricted since it is not satisfied exactly under the Rayleigh–Gans approximation—the error may well be significant for $ka\gtrsim 1\cdot 8$.

Table 1. Forward scattering amplitudes for scattering by various oblate spheroids. Upper entries are exact values; lower entries are those obtained from equation (1). Angles are in degrees. The index indicates the power of ten by which the entry is to be multiplied.

k_0	k_0a	c/a	n_0	θ	$f_V(\theta, 0)$	$f_H(\theta, 0)$	$f_V(53\cdot1, 0)$	$f_H(53\cdot1, 0)$
0·838	0·338	0·65	8·78 +i0·977	16·3	$4\cdot65^{-2}+i6\cdot91^{-3}$	$4\cdot84^{-2}+i8\cdot00^{-3}$	$3\cdot79^{-2}+i6\cdot48^{-3}$	$5\cdot32^{-2}+i1\cdot54^{-2}$
					$4\cdot65^{-2}+i6\cdot91^{-3}$	$4\cdot84^{-2}+i8\cdot00^{-3}$	$3\cdot79^{-2}+i6\cdot49^{-3}$	$5\cdot32^{-2}+i1\cdot54^{-2}$
6·283	0·821	0·875	5·581 + i2·828	16·3	$5\cdot48^{-2}+i6\cdot17^{-2}$	$5\cdot53^{-2}+i6\cdot28^{-2}$	$4\cdot89^{-2}+i5\cdot29^{-2}$	$5\cdot27^{-2}+i6\cdot24^{-2}$
					$5\cdot48^{-2}+i6\cdot17^{-2}$	$5\cdot53^{-2}+i6\cdot28^{-2}$	$4\cdot89^{-2}+i5\cdot30^{-2}$	$5\cdot27^{-2}+i6\cdot24^{-2}$
1·000	1·140	0·80	$1\cdot783+i1\cdot18^{-2}$	25·8	$6\cdot36^{-1}+i1\cdot95^{-1}$	$6\cdot52^{-1}+i2\cdot06^{-1}$	$6\cdot10^{-1}+i1\cdot67^{-1}$	$6\cdot63^{-1}+i2\cdot04^{-1}$
					$6\cdot36^{-1}+i1\cdot95^{-1}$	$6\cdot51^{-1}+i2\cdot06^{-1}$	$6\cdot11^{-1}+i1\cdot69^{-1}$	$6\cdot63^{-1}+i2\cdot04^{-1}$
4·189	1·692	0·65	$1\cdot78\ +i2\cdot4^{-3}$	16·3	$3\cdot69^{-1}+i2\cdot29^{-1}$	$3\cdot72^{-1}+i2\cdot44^{-1}$	$3\cdot85^{-1}+i1\cdot97^{-1}$	$4\cdot15^{-1}+i3\cdot04^{-1}$
					$3\cdot68^{-1}+i2\cdot30^{-1}$	$3\cdot72^{-1}+i2\cdot42^{-1}$	$3\cdot81^{-1}+i2\cdot01^{-1}$	$4\cdot13^{-1}+i2\cdot98^{-1}$

Table 2. Backward scattering amplitudes for scattering by various oblate spheroids. Upper entries are exact values; lower entries are those obtained from equation (2). Angles are in degrees. The index indicates the power of ten by which the entry is to be multiplied.

k_0	k_0a	c/a	n_0	θ	$f_V(\theta, \pi)$	$f_H(\theta, \pi)$	$f_V(53\cdot1, 0)$	$f_H(53\cdot1, 0)$
0·838	0·338	0·65	8·78 +i0·977	16·3	$2\cdot67^{-2}-i4\cdot48^{-3}$	$2\cdot72^{-2}-i5\cdot44^{-3}$	$1\cdot77^{-2}-i4\cdot95^{-3}$	$2\cdot21^{-2}-i1\cdot27^{-2}$
					$2\cdot67^{-2}-i4\cdot48^{-3}$	$2\cdot72^{-2}-i5\cdot43^{-3}$	$1\cdot77^{-2}-i4\cdot94^{-3}$	$2\cdot21^{-2}-i1\cdot27^{-2}$
6·283	0·821	0·875	5·581 + i2·828	16·3	$8\cdot54^{-2}+i3\cdot60^{-2}$	$8\cdot61^{-2}+i3\cdot70^{-2}$	$7\cdot79^{-2}+i2\cdot74^{-2}$	$8\cdot35^{-2}+i3\cdot60^{-2}$
					$8\cdot54^{-2}+i3\cdot60^{-2}$	$8\cdot61^{-2}+i3\cdot70^{-2}$	$7\cdot80^{-2}+i2\cdot75^{-2}$	$8\cdot35^{-2}+i3\cdot60^{-2}$
1·000	1·140	0·80	$1\cdot783+i1\cdot18^{-2}$	25·8	$3\cdot54^{-1}+i1\cdot73^{-1}$	$3\cdot62^{-1}+i1\cdot83^{-1}$	$2\cdot95^{-1}+i1\cdot43^{-1}$	$3\cdot20^{-1}+i1\cdot73^{-1}$
					$3\cdot56^{-1}+i1\cdot74^{-1}$	$3\cdot63^{-1}+i1\cdot83^{-1}$	$2\cdot97^{-1}+i1\cdot44^{-1}$	$3\cdot21^{-1}+i1\cdot73^{-1}$
4·189	1·692	0·65	$1\cdot78\ +i2\cdot4^{-3}$	16·3	$4\cdot96^{-2}+i1\cdot27^{-1}$	$4\cdot86^{-2}+i1\cdot25^{-1}$	$-1\cdot35^{-2}+i5\cdot05^{-2}$	$-2\cdot44^{-2}+i3\cdot73^{-2}$
					$5\cdot34^{-2}+i1\cdot29^{-1}$	$5\cdot19^{-2}+i1\cdot28^{-1}$	$-2\cdot56^{-3}+i5\cdot77^{-2}$	$-1\cdot48^{-2}+i4\cdot43^{-2}$

Table 3. A comparison of the Rayleigh–Gans, Van der Hulst and exact forward scattering amplitudes for scattering by various oblate spheroids, $a/c = 2{\cdot}0$, $k_0 = 1{\cdot}0$. The index indicates the power of ten by which the entry is to be multiplied.

Spheroid		Scattering Amplitudes					
		All θ	$\theta = 0$		$\theta = \pi/2$		
a	n_0	Rayleigh–Gans	Van der Hulst	Exact	Van der Hulst	Exact (V)	Exact (H)
8·0	1·01	$1{\cdot}72^{0}$	$1{\cdot}71^{0}+i5{\cdot}13^{-2}$	$1{\cdot}71^{0}+i5{\cdot}14^{-2}$	$1{\cdot}70^{0}+i1{\cdot}02^{-1}$	$1{\cdot}73^{0}+i8{\cdot}41^{-2}$	$1{\cdot}73^{0}+i8{\cdot}96^{-2}$
8·0	1·10	$1{\cdot}79^{1}$	$1{\cdot}60^{1}+i4{\cdot}94^{0}$	$1{\cdot}64^{1}+i5{\cdot}06^{0}$	$1{\cdot}31^{1}+i8{\cdot}87^{0}$	$1{\cdot}62^{1}+i8{\cdot}85^{0}$	$1{\cdot}64^{1}+i9{\cdot}57^{0}$
1·0	1·33	$1{\cdot}28^{-1}$	$1{\cdot}09^{-1}+i1{\cdot}35^{-2}$	$1{\cdot}18^{-1}+i7{\cdot}67^{-3}$	$1{\cdot}05^{-1}+i2{\cdot}66^{-2}$	$9{\cdot}79^{-2}+i4{\cdot}41^{-3}$	$1{\cdot}19^{-1}+i6{\cdot}79^{-3}$
5·0	1·33	$1{\cdot}60^{1}$	$1{\cdot}04^{1}+i7{\cdot}31^{0}$	$1{\cdot}08^{1}+i7{\cdot}46^{0}$	$3{\cdot}56^{1}+i9{\cdot}13^{0}$	$8{\cdot}71^{0}+i1{\cdot}17^{1}$	$8{\cdot}33^{0}+i1{\cdot}32^{1}$
10·0	1·33	$1{\cdot}28^{2}$	$2{\cdot}85^{1}+i7{\cdot}30^{1}$	$3{\cdot}08^{1}+i8{\cdot}18^{1}$	$-6{\cdot}84^{0}+i2{\cdot}27^{1}$	$-2{\cdot}13^{1}+i5{\cdot}25^{1}$	$-1{\cdot}49^{1}+i5{\cdot}99^{1}$

The above approximations all assume that the refractive index is close to unity. However many calculations of the exact amplitudes, using the Fredholm integral equation method (Holt *et al* 1976) have shown that the forward and backward scattering rules apply to a much larger range of refractive index than has been established here theoretically. The general conclusion, that the forward rule applies to a wider range of ka than the backward, is also observed, however, in the empirical evidence. In tables 1 and 2 we give some examples of the comparison between the exact amplitudes and those calculated via the scattering rules. The better agreement of the forward amplitudes should be noted. As ka increases the backward amplitudes for 90° orientation change sign, and this appears to be the reason for the failure of equation (2) as ka increases. The range of ka for which equation (2) is valid appears to be a little smaller than suggested by the Rayleigh–Gans approximation.

As far as we are aware this is the first application of the Van der Hulst approximation to spheroids. Since it reduces to the Rayleigh–Gans approximation for ka small, and since the latter is frequently used at present to calculate scattering cross sections it seems worthwhile to compare the two approximations. In table 3 we give the comparison for oblate spheroids for a range of real refractive index. It will be seen that for incidence along the axis of symmetry, the Van der Hulst approximation gives a good estimate of the real part of the scattering amplitude—much better than the Rayleigh–Gans approximation—for the case considered. It is not so successful in estimating the imaginary part (which the Rayleigh–Gans estimates to be zero). However for 90° incidence the Van der Hulst approximation is much less successful, and the Rayleigh–Gans may be preferable. It would seem that the reason may be that for 0° incidence (along the axis of symmetry), not only is the path in the spheroid shortest, but the angle of incidence at the surface of the spheroid will be close to 90° for much of the geometrical cross section. In addition there is a symmetry which means that the exact amplitudes are identical for both polarisations. Hence the assumptions of the Van der Hulst approximation should be good. At 90° incidence none of these factors occurs. This explanation is also supported by the case of prolate spheroids where the shortest path through the spheroid will be for 90° incidence. Here sample calculations have shown the Van der Hulst approximation is indeed best for 90° incidence. However the degree of agreement is not as good (as it is for 0° incidence for oblate spheroids), and this is probably due to the lack of symmetry of the geometrical cross section.

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